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ON AN EXAMPLE OF HANF

Spaces with infinitely many isolated points, such that n isolated points can be left out, but not 1, 2, ..., n-1

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Introduction

W. Hanf proved in [3] the existence of a zerodimensional compact Hausdorff space with infinitely many isolated points, from which n isolated points, but not 1, 2, ... or n-1 can be removed without changing the space topologically. The construction was given in terms of Boolean algebras.

Later, in [2], R. Engelking constructed in a purely topological way a zerodimensional compact Hausdorff space with infinitely many isolated points with the much weaker property that no isolated point can be removed without changing the space topologically.

In this note we shall describe a very simple, purely topological example with the properties of the example of Hanf. Moreover a partial solution will be given to the problem,for which Hausdorff spaces X no points of the Alexandroff duplicate $X \cup X'$ (as described in [2]) can be left out.

§1. Main Example

$\beta\mathbb{N}$ denotes the Čech-Stone compactification of $\mathbb{N}$. $\mathbb{N}_1, \mathbb{N}_2, \ldots, \mathbb{N}_{n-1}$ are n-1 disjoint copies of $\mathbb{N}$. The elements of $\mathbb{N}_i$ are denoted by $1_i, 2_i, 3_i, \ldots$.

Let X be the set $\beta\mathbb{N} \cup \mathbb{N}_1 \cup \ldots \cup \mathbb{N}_{n-1}$, supplied with the topology generated by the following sets:

a) all singletons $\{k_i\}$ and $\{k\}$ for $k \in \mathbb{N}$, $i < n$

b) all sets $O \cup \{k_i : k \in O \cap \mathbb{N},\ i = 1, 2, \ldots, n-1,\ k_i \neq x^{(1)}, \ldots, x^{(m)}\}$ for O an open subset of $\beta\mathbb{N}$ and $\{x^{(1)},\ldots,x^{(m)}\}$ an arbitrary finite subset of $\mathbb{N} \cup \mathbb{N}_1 \cup \ldots \cup \mathbb{N}_{n-1}$.

X will be the example. Notice that X is compact and that $\mathbb{N} \cup \mathbb{N}_1 \cup \ldots \cup \mathbb{N}_{n-1}$ is the set of isolated points of X. For $n = 1$, $X = \beta\mathbb{N}$ and for $n = 2$, X is a subset of the Alexandroff duplicate of $\beta\mathbb{N}$. Next we define the map $\pi: X \to \beta\mathbb{N}$ by $\pi x = x$ for $x \in \beta\mathbb{N}$ and $\pi k_i = k$ for $i < n$ and $k \in \mathbb{N}$. Now let f be a homeomorphism from X into, not necessarily onto itself.

LEMMA 1.1. For at most finitely many pairs (k,i) with $k \in \mathbb{N}$ and $i < n$ $\pi f k_i \neq \pi f k$.

Proof. Suppose for infinitely many pairs (k,i), $\pi f k_i \neq \pi f k$. Then for some $i < n$ there are infinitely many $k \in \mathbb{N}$ such that $\pi f k_i \neq \pi f k$. We may even choose an infinite set $A \subset \mathbb{N}$ such that $\pi f A \cap \pi f\{k_i : k \in A\} = \emptyset$. Disjoint subsets of $\mathbb{N}$ have disjoint closures in $\beta\mathbb{N}$, so $\pi f A$ and $\pi f\{k_i : k \in A\}$ have disjoint sets of accumulationpoints. By compactness, these sets of accumulationpoints are not empty so they are different. However, it is easily seen that for each subset Y of X, πY and Y have the same accumulationpoints. If we choose $Y = \{k_i : k \in A\}$ and keep in mind that f is a homeomorphism, then it follows that fA and $f\{k_i : k \in A\}$ have the same accumulationpoints. Because the same holds for $\pi f A$ and $\pi f\{k_i : k \in A\}$, a contradiction follows. □

THEOREM 1.2. X is a zerodimensional compact Hausdorff space with countably many isolated points, from which n isolated points, but not 1, 2, ..., n-1 can be removed by a homeomorphism.

<u>Proof</u>. Let $f\colon X \to X$ be any homeomorphism. First we have to show that $|X\backslash fX| \neq 1, 2, \ldots, n-1$. We may suppose that $|X\backslash fX|$ is finite. $\beta\mathbb{N}\backslash\mathbb{N}$ is the set of non-isolated points, so points of $\beta\mathbb{N}\backslash\mathbb{N}$ are mapped onto points of $\beta\mathbb{N}\backslash\mathbb{N}$ and points of $\mathbb{N} \cup \mathbb{N}_1 \cup \ldots \cup \mathbb{N}_{n-1}$ are mapped onto points of $\mathbb{N} \cup \mathbb{N}_1 \cup \ldots \cup \mathbb{N}_{n-1}$. Thus $f(\beta\mathbb{N}\backslash\mathbb{N})$ equals $\beta\mathbb{N}\backslash\mathbb{N}$, maybe with the exception of a finite number of (non-isolated) points which are member of $\beta\mathbb{N}\backslash\mathbb{N}$ but not of $f(\beta\mathbb{N}\backslash\mathbb{N})$. Since $f(\beta\mathbb{N}\backslash\mathbb{N})$ is compact, it follows that $f(\beta\mathbb{N}\backslash\mathbb{N}) = \beta\mathbb{N}\backslash\mathbb{N}$. Now suppose that $\{x^{(1)},\ldots,x^{(k)}\}$ is the set of points x for which $\pi f x \neq \pi f \pi x$. Then for some finite subset B of $\mathbb{N}$, $\{x^{(1)},\ldots,x^{(k)}\} \subseteq \pi^{-1}B$. The choice of B and the previous remarks yield that $x \in f(X\backslash\pi^{-1}B)$ is equivalent to $\pi^{-1}\pi x \subseteq f(X\backslash\pi^{-1}B)$. Thus $X\backslash f(X\backslash\pi^{-1}B)$ can be written as $\pi^{-1}C$ with C a finite subset of $\mathbb{N}$. Hence $|X\backslash f(X\backslash\pi^{-1}B)|$ is a multiple of n. Since $|f\pi^{-1}B| = |\pi^{-1}B|$ is also a multiple of n, this is also true for $|X\backslash fX|$, so $|X\backslash fX| \neq 1, 2, \ldots, n-1$.

The only thing that still has to be done is to find a homeomorphism that removes exactly n points from X. Define $gk = k+1$ for $k \in \mathbb{N}$. This map can be extended to a homeomorphism g' from $\beta\mathbb{N}$ onto $\beta\mathbb{N}\backslash\{1\}$. Moreover, if we define $g'(k_i) = (k+1)_i$, we obtain a homeomorphism from X onto $X\backslash\{1,1_1,1_2,\ldots,1_{n-1}\}$. □

§2. For which Hausdorff spaces X no point of the Alexandroff duplicate can be left out by a homeomorphism?

For X is a Hausdorff space, let X' be a set-theoretical copy of X and let $x \to x'$ be a fixed bijection from X to X'. Take for a base for the topology all sets $\{x'\}$ for $x \in X$ and all sets $(O \cup O') \setminus \{x_1', x_2', \ldots, x_n'\}$ for O open in X and $x_1, \ldots, x_n$ arbitrary points of X. The set $X \cup X'$, supplied with this topology, is called the Alexandroff duplicate of X. Alexandroff defined this space for a circle in [1] as an example of a compact first countable Hausdorff space wit $2^{\aleph_0}$ isolated points. Engelking generalized the construction to arbitrary Hausdorff spaces in [2]. If X is compact, then $X \cup X'$ is also compact. Define $\pi: X \cup X' \to X$ by $\pi(x') = x$ and $\pi(x) = x$. It is easily seen that for each $A \subseteq X \cup X'$, πA and A have the same accumulationpoints.

THEOREM 2.1. <u>If X contains a convergent sequence or an infinite closed discrete subset, in particular if X is first countable, then each isolated point of $X \cup X'$ can be left out.</u>

<u>Proof</u>. Let $(x_i)_i$ be a convergent sequence or a sequence without limitpoints in X. Then the map which sends x_i' onto x_{i+1}' for $i \in \mathbb{N}$ and which is identity on all other points, is a homeomorphism from $X \cup X'$ onto $(X \cup X') \setminus \{x_1'\}$. This proves the theorem, because each two isolated points can be interchanged by a homeomorphism which is identity on the other points. □

THEOREM 2.2. <u>If every sequence in X has at least two limitpoints and quasicomponents are points and X contains no (or at most finitely many) isolated points, then no point of $X \cup X'$ can be left out.</u>

<u>Proof</u>. Suppose $f: X \cup X' \to X \cup X'$ is a homeomorphism that removes one point, x_0, from X. X has no isolated points, so f maps X' into X' and X into X, thus f has an inverse $g: X \setminus \{x_0\} \to X$. Add a new isolated point p to $X \cup X'$. Extend g to a homeomorphism $g': (X \cup X') \setminus \{x_0\} \to$ $\to X \cup X' \cup \{p\}$ as follows:

$g'(x') = y'$ with $y = g(x)$ for $x \neq x_0$

$g'(x'_0) = p.$

Then $(g' \circ f)^{-1} : X \cup X' \cup \{p\} \to X \cup X'$ is a homeomorphism that leaves out exactly one isolated point from a space which is apparently homeomorphic to $X \cup X'$ and $(g' \circ f)^{-1}$ is identity on the non-isolated points.

Now suppose that f is a homeomorphism that removes one point x'_0 from X'. Let g be the inverse of f on X and g' the extension of g over $X \cup X'$ defined by $g'(x') = y'$ for $y = g(x)$. Then $g' \circ f$ also is a homeomorphism on $X \cup X'$ that leaves out exactly one isolated point and is identity on X.

So we may assume that f leaves out a point x'_0 of X' and is identity on X. Consider the sequence $S = \{x'_0, f(x'_0), f^2(x'_0), \ldots\}$. Observe that all terms are different. By choice of X, πS (thus S too) has (at least) two accumulationpoints. All accumulationpoints are in X. Call these two accumulationpoints x_1 and x_2. Because the quasicomponents of X are points, X is the topological sum of two disjoint clopen subsets A_1 and A_2, such that $x_1 \in A_1$ and $x_2 \in A_2$. This partition induces a partition of S into two infinite subsets S_1 and S_2 : $S_i = \{x \in S : \pi x \in A_i\}$. Note that each accumulationpoint of S_i is in A_i. Now $S_1 \cap f^{-1}S_2$ is infinite too. Hence $S_1 \cap f^{-1}S_2$ has some accumulationpoints, which are all in the closed set A_1. The accumulationpoints of $f(S_1 \cap f^{-1}S_2)$ are all in the closed set A_2, because $f(S_1 \cap f^{-1}S_2) \subset S_2$. But they are also elements of $fA_1 = A_1$. So the assumptiom that f removes one point leads to a contradiction. Actually, the same proof yields that f cannot remove any non-empty subset of X' without removing some part of X too. □

COROLLARY 2.3. <u>No point and no set of isolated points can be removed from $(\beta N \setminus N) \cup (\beta N \setminus N)'$ by a homeomorphism.</u>

THEOREM 2.4. <u>If every sequence in X has at least two limitpoints and if X is extremally disconnected, then no point of $X \cup X'$ can be left out.</u>

The proof goes very much like the proof of theorem 1.2. In particular, we need the lemma:

LEMMA 2.5. If X fulfils the conditions of the theorem and if f is a homeomorphism from $X \cup X'$ into $X \cup X'$, then $\pi fx = \pi fx'$ for almost every $x \in X$.

Proof of the lemma. Suppose $\pi fx \neq \pi fx'$ for infinitely many $x \in X$. Then there exists an infinite set $A \subset X$ such that $\pi fA \cap \pi fA' = \emptyset$. In the present case, we cannot conclude that fA and fA' have different accumulationpoints. But we shall find an infinite set $D \subset A$ such that $\pi fD \cup \pi fD'$ is discrete and then we are through (because of extreme disconnectedness of X). First we choose an infinite discrete subset Y of πfA (each infinite Hausdorff space has an infinite discrete subset) and choose $B \subseteq A$ such that $\pi fB = Y$. Similarly we can find an infinite set $C \subset B$ such that $\pi fC'$ is discrete. Now it is still possible that $\pi fC' \cup \pi fC$ is not discrete, so we shall carry out another, last step. We may assume that πf is one-to-one on $C \cup C'$. Under this assumption there exists a canonical one-to-one correspondence F between πfC and $\pi fC'$: $y = \pi fc \Leftrightarrow Fy = \pi fc'$. Now choose $y_1 \in \pi fC$ and a closed neighborhood V_1 in X for y_1 such that $V_1 \cap \pi fC = \{y_1\}$ and $\pi fC' \setminus V_1$ is infinite and contains Fy_1. Define $D_1 = \{c \in C : \pi fc \notin V_1 \text{ and } \pi fc' \notin V_1\}$. D_1 is infinite and $Fy_1 \in \pi fD_1'$. Choose a closed neighborhood W_1 in X for Fy_1, such that $\pi fD_1' \cap W_1 = \{Fy_1\}$ and $\pi fD_1 \setminus W_1$ is infinite. Choose W_1 disjoint from V_1. Define $C_1 = \{c \in D_1 : \pi fc \notin W_1 \text{ and } \pi fc' \notin W_1\}$. C_1 is infinite. Choose $y_2 \in \pi fC_1$ and a closed neighborhood V_2 of y_2, such that $V_2 \cap (V_1 \cup W_1) = \emptyset$, $V_2 \cap \pi fC_1 = \{y_2\}$, $\pi fC_1' \setminus V_2$ is infinite and $Fy_2 \notin V_2$. Define $D_2 = \{c \in C_1 : \pi fc \notin V_2 \text{ and } \pi fc' \notin V_2\}$. D_2 is infinite and $Fy_2 \in \pi fD_2'$. Choose a closed neighborhood W_2 of Fy_2, such that $W_2 \cap (V_1 \cup W_1 \cup V_2) = \emptyset$, $W_2 \cap \pi fD_2' = \{Fy_2\}$ and $\pi fD_2 \setminus W_2$ is infinite. Define $C_2 = \{c \in D_2 : \pi fc \notin W_2 \text{ and } \pi fc' \notin W_2\}$. Again C_2 is infinite. Now it should be clear how to repeat the procedure infinitely many times. The result is an infinite discrete set $\{y_i\}_i \cup \{Fy_i\}_i$ of the form $\pi fD \cup \pi fD'$. Since X is extremally disconnected, $\pi fD'$ and πfD have different (in fact disjoint) closures, whilst they should have the same, non-void set of accumulationpoints. For f is a homeomorphism and D and D' share all accumulationpoints. Contradiction.

□

Proof of the theorem. This is an obvious modification of the proof of theorem 1.2. □

COROLLARY 2.6. *No point of $\beta\mathbb{N} \cup (\beta\mathbb{N})'$ can be removed by a homeomorphism.*

Proof. Apply compactness of $\beta\mathbb{N}$ for the non-isolated points of $\beta\mathbb{N}$. □

REMARK 2.7. Lemma 2.5 is not generally true if X is not extremally disconnected. Choose e.g. $X = \beta\mathbb{N} \cup (\beta\mathbb{N})'$ and define $fn = n'$, $fn' = n$ and f is identity on all other points of $X \cup X'$. Of course this space is not a counter example to the theorem.

REMARK 2.8. Theorems 2.2 and 2.4 can be generalized by taking n-1 copies $X_1, \ldots, X_{n-1}$ of X instead of one. Take as a base for the topology all sets $\{x_i\}$ for $x \in X$ and $i < n$, and all sets $(O \cup O_1 \cup \ldots \cup O_{n-1}) \setminus \{p^{(1)}, \ldots, p^{(k)}\}$, O open in X, $\{p^{(1)}, \ldots, p^{(k)}\}$ an arbitrary finite subset of $X_1 \cup \ldots \cup X_{n-1}$. Then we can prove that one cannot leave out i points for $i = 1, 2, \ldots, n-1$. Moreover, if one point can be left out from X, then multiples of n can be left out from $X \cup X_1 \cup \ldots \cup X_{n-1}$. Thus we see that $\beta\mathbb{N} \cup \beta\mathbb{N}_1 \cup \ldots \cup \beta\mathbb{N}_{n-1}$ also has all properties mentioned in theorem 1.2, except the number of isolated points.

REMARK 2.9. In the proof of theorem 2.1 we only use the property that $X \cup X'$ has a sequence of isolated points which either converges or has no limitpoints at all. Thus if a space Y has these properties, then Y is homeomorphic to $Y \setminus \{y\}$ for any isolated point $y \in Y$. Moreover, Hanf mentiones in [3] a result of R.L. Vaught that every zerodimensional separable compact Hausdorff space X with infinitely many isolated points is homeomorphic to $X \setminus \{x\}$ for every isolated point x of X.

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